A Multipole Expansions Method for Acoustic Wave Propagation in Vocal Tract

F. Seydou$^{1,2}$, T. Seppälä$^1$ and O. M. Ramahi$^{2,3,4}$

$^1$Department of Electrical and Information Engineering
University of Oulu, Finland

$^2$Mechanical Engineering Department, $^3$Electrical and Computer Engineering, and
$^4$CALCE Electronic Products and Systems Center
A. James Clark School of Engineering, University of Maryland
College Park, MD 20742, USA

1. Introduction

The main purpose of this paper is to investigate the problem of wave motion in a vocal tract. We try and solve the problem with use of Helmholtz equation, one of the most important mathematical models which is used to describe the time harmonic behavior of various vibration and wave propagation phenomena. The motivation of our work is to understand the main characteristics of wave propagation in the vocal tract, which is modeled as a series of 8 concatenated tube. Some examples of research applications include human speech production, speech recognition and speaker identification. Some of the characteristic quantities to be calculated in these problems include scattering amplitudes, transmission and reflection coefficients and resonance frequencies.

Often the methods that are used are finite element type (cf. [4]), volume integral (cf. [3], or domain decomposition type (cf. [5]). While these methods give fairly satisfactory results, they are not for the case of very high frequencies. From a computational point of view, it is highly desirable to obtain a closed form solution of the scattering problem since for many applications we need high frequency solutions. In this paper we use multipole expansions to derive a close form solution of the problem.

2. The mathematical model

Our approach to modeling sound transmission in the vocal tract is through the use of concatenated lossless acoustic tubes as depicted in figure 1. For time-harmonic solution of the wave equation we can deduce (cf. [5]) that we have to solve the following problem

\[ (\nabla^2 + \omega^2)u = f \quad \text{in} \ D, \]

\[ \alpha \frac{\partial u}{\partial n} + \beta u = g \quad \text{on} \ D, \]

where \( u \) satisfies radiation condition, \( \omega \) is the frequency, \( \alpha \) and \( \beta \) are given constants, \( f \) and \( g \) are given functions, \( D \) is the domain describing the vocal tract, \( \partial \) its boundary and \( n \) the unit outward normal.
To tackle the problem, let us divide the boundary by NG uniformly distributed grid points. Next, we take the grid points to be centers of tiny non-overlapping circles $C_j$ with centers $O_j$, $j = 1, \cdots, NG$, and a constant radius $r_0$, as shown in Figure 2. Finally, the scattering problem is regarded as a multiple scattering of acoustic waves by NG infinitely long cylinders with cross sections $C_j$. Our claim is that the solution of the multiple scattering problem is a good approximation of the solution of the model scattering problem. To show this, we use multipole expansion method to obtain a closed form solution of the multiple scattering problem.

3. The Multipole Expansions Solution

Our approach is based on the assumption that the boundary condition is satisfied in a very close neighborhood of the boundary and thus on the tiny circles $C_j$. In the sequel, without loss of generality we assume that $\alpha = 0$, $\beta = 1$, $f = 0$ and $g$ is a plane wave. It can be shown (cf. [2]) that, for a point $x \in D \setminus \bigcup_{j=1}^{NG} C_j$, $u$ has the form

$$u(x) = \sum_{j=1}^{NG} \sum_{m=-\infty}^{\infty} b_{jm} H_0^{(1)}(k_j r_j(x)) e^{i\theta(x)}$$

where $r_j(x)$ and $\theta(x)$ are the polar coordinates of a point $x \in D$ in the coordinate system $X_jO_jY_j$.

To use the boundary conditions we must take a reference coordinate system $X_0O_0Y_0$, and the field expressed in $X_jO_jY_j$ coordinate must be translated to $X_0O_0Y_0$ coordinate. From the addition formula, we have

$$u_{\delta}(x) + u'(x) = \sum_{\ell=-\infty}^{\infty} \left( c_{\ell} + a_{\ell} H_0^{(1)}(k_{\ell} r_j(x)) \right) e^{i\theta_j(x)} \quad (2.1)$$

where
\[ c_{i,a} = \sum_{j=1}^{N_G} \sum_{m=-\infty}^{\infty} \tilde{H}_{\nu-m}^{(1)}(k_0 \sigma_j) e^{-i(k_0 - \sigma_j) \Delta \phi} b_{j,m} - e^{-i\Delta \phi_0 - i\pi/2} \] (2.2)

The boundary condition implies that, for \( l = 1, \cdots, N_G \),
\[ c_{i,a} J_l(k_0 \sigma_j) + b_{l,m} H_l^{(1)}(k_0 \sigma_j) = 0 \]
which we write as
\[ c_{i,a} = S_l b_{l,a} \] (2.3)
where \( S_l \) is a diagonal matrix independent of \( l \) given by
\[ S_l = \frac{H_l^{(1)}(k_0 \sigma_j)}{J_l(k_0 \sigma_j)} \]
Equations (2.2) and (2.3) give the following infinite linear system of equations for \( l = 1, \cdots, N_G \),
\[ S_l b_{l,a} - \sum_{j=1}^{N_G} \sum_{m=-\infty}^{\infty} \tilde{H}_{\nu-m}^{(1)}(k_0 \sigma_j) e^{-i(k_0 - \sigma_j) \Delta \phi} b_{j,m} = e^{i\Delta \phi_0 + i\pi/2} \]
Of course the sum has to be truncated, at a number \( N_0 \) to get a finite system. By solving the system we obtain the coefficients \( b_{j,m} \) for \( j = 1, \cdots, NG \) and \( -N \leq m \leq N \). Then the total field \( u_0 + u' \) can be recovered from (2.1).

![Figure 2: Discretization of the tube.](image)

**4. Numerical results**

We consider the case of scattering by a circle where the exact solution is well known. The result is reported in Figure 3 where an excellent match of the two methods can be found.
5. Conclusion

We have developed a multipole expansions method for solving the acoustic wave propagation phenomena in vocal tract. For testing the method we show the numerical results for the case of a disc where the solution is well known. Our approach shows excellent results. In the future we intend to extend the method for the three dimensional case and to related inverse problems.

References